**L E C T U R E 14**

**Random variables. The law of distribution of a discrete random variable**

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself. For instance, in tossing dice we are often interested in the sum of the two dice and are not really concerned about the separate values of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1). Also, in coin flipping, we may be interested in the total number of heads that occur and not care at all about the actual head-tail sequence that result. These quantities of interest, or more formally, these real-valued functions defined on the outcome space, are known as *random variables*.

*A random variable* is understood as a variable which as result of a trial takes one of the possible set of its values (which namely – it is not beforehand known).

We denote random variables by capital letters of Latin alphabet *X, Y, Z*, …, and their values – by the corresponding small letters *x, y, z*, ….

*Example.* The number of the born boys among hundred newborns is a random variable which has the following possible values: 0, 1, 2, …, 100.

*Example.* The distance which will be flied by a shell at shot by a gun is a random variable. Really, the distance depends not only on installation of a sight, but also from many other reasons (force and direction of wind, temperature, etc.) which cannot be completely taken into account. Possible values of this variable belong to some interval (*a, b*).

*Example.* Suppose that our experiment consists of tossing 3 coins. If we let *Y* denote the number of heads appearing, then *Y* is a random variable taking on one of the values 0, 1, 2, 3 with respective probabilities

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*A discrete* random variable is a random variable which takes on separate, isolated possible values with certain probabilities. The number of possible values of a discrete random variable can be finite or infinite.

For a discrete random variable *X*, we define the *probability mass function p(a)* of *X* by

*p(a) = P(X = a)*

*A continuous* random variable is a random variable which can take all values from some finite or infinite interval. Obviously, the number of possible values of a continuous random variable is infinite.

The most full, exhaustive description of a random variable is its law of distribution.

Any ratio establishing connection between possible values of a random variable and probabilities corresponding to them refers to as the *law of distribution* of the random variable.

About a random variable speak that it «is distributed» under the given law of distribution or «subordinated» to this law of distribution.

For a discrete random variable the law of distribution can be set as a table, analytically (as a formula) and graphically.

The elementary form of assignment of the law of distribution of a discrete random variable *X* is a table (matrix) in which all possible values of a random variable and the probabilities corresponding to them are listed in ascending order, i.e.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *Х* | *x1* | *х2* | *…* | *xn* |
| *р* | *р1* | *p2* | *…* | *pn* |

Such a table is called *the series of distribution* of a discrete random variable.

The events *X = x1, X = x2, …, X = xn*, consisting in that as a result of trial the random variable *X* will take on values *x1, x2, …, xn* respectively, are incompatible and uniquely possible (because in the table all possible values of a random variable are listed), i.e. form a complete group. Hence, the sum of probabilities is equal to 1. Thus, for every discrete random variable

*р1 + р2 +…+ рn = 1*

(This unit is somehow *distributed* between values of a random variable, therefore from here the term "distribution").

A series of distribution can be represented graphically if values of a random variable are postponed on the axis of abscissas, and on the axis of ordinates – their corresponding probabilities. Connecting the received points forms a broken line named a *polygon of distribution of probabilities*.

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*Example*. 100 tickets of a monetary lottery are released. One prize in 50 roubles and ten prizes on 1 rouble are played. Find the law of distribution of a random variable *X* – cost of a possible prize for an owner of one lottery ticket.

*Solution*: Write the possible values of *Х: х1 = 50, х2 = 1, х3 = 0.* The probabilities of these possible values are those: *р1 = 0,01; р2 = 0,1; р3 = 1 – (р1 + р2) = 0,89.*

Let's write the required law of distribution:

|  |  |  |  |
| --- | --- | --- | --- |
| *Х* | *50* | *1* | *0* |
| *р* | *0.01* | *0.1* | *0.89* |

**Mathematical operations over random variables**

Two random variables are *independent* if the law of distribution of one of them does not vary from that which possible values were taken on by another variable. So, if a discrete random variable *X* can take on values *xi (i = 1, 2, …, n),* and a random variable *Y* – values *yj (j = 1, 2, .., m)* then the independence of the discrete random variables *X* and *Y* means the independence of the events *X = xi* and *Y = yj* for all *i = 1, 2, .., n* and *j = 1, 2, .., m*. Otherwise, the random variables are *dependent.*

*The m-th degree of a random variable X*, i.e. *X m* is the random variable which takes on values *xim* with the same probabilities *pi (i = 1, 2, .., n)*.

*The sum* (*the difference* or *the product*) of random variables *X* and *Y* is the random variable which takes on all possible values of kind *xi + yj* (*xi – yj* or *xi* \* *yj*) where *i = 1, 2, …, n; j = 1, 2, …, m* with the probabilities *pij* that the random variable *X* will take on the value *xi*, and *Y* – the value *yj*:

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**(Mathematical) expectation of a discrete random variable**

One of the most important concepts in probability theory is the expectation of a random variable. If *X* is a discrete random variable having a probability mass function *p(x)*, the *(mathematical) expectation* (*the expected value* or *the mean*) of *X*, denoted by *M(X),* is defined by

М(X) = xi pi.

*Property 1.* The mathematical expectation of a constant is equal to the constant:

*M(C) = C*

*Property 2.* A constant multiplier can be taken out for a sign of mathematical expectation, i.e. *M(kX) = kM(X)*

*Property 3.* The mathematical expectation of the algebraic sum of finitely many random variables is equal to the sum of their mathematical expectations.

*Property 4*. The mathematical expectation of the product of finitely many mutually independent random variables is equal to the product of their mathematical expectations.

*Property 5*. The mathematical expectation of deviation of a random variable from its mathematical expectation is equal to zero:

*M[X – M(X)] = 0*

**Dispersion of a discrete random variable**

Although *M(X)* yields the weighted average of the possible values of *X*, it does not tell us anything about the variation, or spread, of these values. For instance, although random variables *W, Y*, and *Z*, having probability mass functions determined by

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All have the same expectation – namely, 0 – there is much greater spread in the possible value of *Y* than in those of *W* (which is a constant) and in the possible values of *Z* than in those of *Y*.

As we expect *X* to take on values around its mean *M(X)*, it would appear that a reasonable way of measuring the possible variation of *X* would be to look at how far apart *X* would be from its mean on the average. In practice it is often required to estimate the dispersion (variation) of possible values of a random variable around of its average value. For example, in artillery it is important to know as far as shells will concentrically lie near to the target which should be struck.

One possible way to measure this would be to consider the quantity *M(|X – a|)*, where *a = M(X)*. However, it turns out to be mathematically inconvenient to deal with this quantity, and so a more tractable quantity is usually considered – namely, the expectation of the square of the difference between *X* and its mean. We thus have the following definition:

If *X* is a random variable with expectation *M(X)*, then *the dispersion (variance)* of *X*, denoted by *D(X)*, is defined by

*D(X) = M[X – M(X)]2*.

**Remark.** Analogous to the mean being the center of gravity of a distribution of mass, the dispersion (variance) represents, in the terminology of mechanics, *the moment of inertia*.

*The mean square deviation* (*the standard deviation*) ****** of a random variable *X* is the arithmetic value of the square root of its dispersion:

*Property 1*. The dispersion of a constant is equal to zero: *D(C) = 0.*

*Property 2*. A constant multiplier can be taken out from the argument of the dispersion involving it in square:

*D(kX) = k 2 D(X)*

*Property 3*. The dispersion of a random variable is equal to the difference between the mathematical expectation of the square of the random variable and the square of its mathematical expectation:

*D(X) = M(X 2) – [M(X)]2*

*Property 4.* The dispersion of the algebraic sum of finitely many mutually independent random variables is equal to the sum of their dispersions

Observe that *the dispersion of both the sum and the difference of independent random variables X and Y is equal to the sum of their dispersions*, i.e.

*D(X + Y) = D(X – Y) = D(X) + D(Y).*

The mathematical expectation, the dispersion and the mean square deviation are *numerical characteristics* of a random variable.

Екi мергеннiңцентрден ауытқуларының үлестiрiм заңдары белгiлi:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Х | 1 | 2 | 3 | 4 |  | У | 0 | 1 | 3 | 5 |
| p | 0,4 | 0,3 | 0,2 | 0,1 |  | p | 0,1 | 0,45 | 0,35 | 0,1 |

Осы екi мергеннiң бiреуiн ғана жоғары деңгейдегi жарысқа таңдап алу керек. Ол үшiн екi кездейсоқ шама математикалық үміттерiн салыстырып қайсысы аз мәнге ие болса, соны таңдап аламыз (себебi, центрден аз ауытқыған). Математикалық үміттерiн есептейiк:

М(Х)=1⋅0,4+2⋅0,3+3⋅0,2+4⋅0,1=2;

M(У)=0⋅0,1+1⋅0,45+3⋅0,35+5⋅0,1=2.

Мысалдағы екi мергеннiң де дисперсияларын есептейiк. Ол үшiн Х2 және У2 кездейсоқ шамалардың үлестiрiм заңын жазып алайық:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Х2 | 1 | 4 | 9 | 16 |  | У2 | 0 | 1 | 9 | 25 |
| p | 0,4 | 0,3 | 0,2 | 0,1 |  | p | 0,1 | 0,45 | 0,35 | 0,1 |

М(Х2) пен М(У2) есептейiк:

М(Х2)=1⋅0,4+4⋅0,3+9⋅0,2+16⋅0,1=5;

M(У2)=0⋅0,1+1⋅0,45+9⋅0,35+25⋅0,1=6,1.

Ендi дисперсияларын есептейiк:

D(X)=М(Х2)-[М(Х)]2=5-2=3; D(У)=М(У2)-[М(У)]2=6,1-2=4,1.

Екi кездейсоқ шаманың дисперсиялары әртүрлi: D(X)=3 және D(У)=4,1. Бiрiншi мергеннiң көрсеткiштерi екiншi мерген көрсеткiштерiне қарағанда орта шамадан шашырауы аз. Басқаша айтсақ бiрiншi мергеннiң көрсеткiштерi орта шама маңайында, екiншiге қарағанда, көбiрек шоғырланған. Бұл қортынды жоғары деңгейдегi жарысқа бiрiншi мергеннiң таңдалуына негiз болады.

**Distribution function of a random variable**

For a random variable *X,* the function *F* defined by

*F(x) = P(X < x),* - ∞ *< x <* + ∞

is called *the cumulative distribution function* or, more simply, *the distribution function* of *X*. Thus the distribution function specifies, for all real values *x*, the probability that the random variable is less than *x*.

Sometimes the distribution function *F(x)* is said to be the *integral function of distribution* or the *integral law of distribution*.

Geometrically the distribution function is interpreted as the probability that a random variable *X* will hit to the left from a given point *x*.

**Properties of a distribution function**

1. The distribution function of a random variable is a non-negative function taking on values between 0 and 1: 0≤ F (x) ≤ 1.

2. The distribution function of a random variable is a non-decreasing function for the entire numerical axis, i.e. if *x1 < x2* then *F(x1)* < *F(x2)*.

3. F (x) → 0 when х → - ∞; F (x) → 1 when х → + ∞.

4. The probability of hit of a random variable in an interval [*x1, x2*) (including *x1*) is equal to the increment of its distribution function on this interval, i.e.

*P(x1* < *X < x2) = F(x2) – F(x1)*.

*The probability density* (*distribution density* or simply *density*) *f (x)* of a continuous random variable *X* is the derivative of its distribution function, i.e.

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Sometimes the probability density is said to be the *differential function* or the *differential law of distribution*.

The graph of probability density *f(x)* is said to be *distribution curve*.

**Properties of probability density**

1. The probability density is a non-negative function, i.e. *******.*

2. The probability of hit of a continuous random variable in an interval [*x1, x2*] is equal to the definite integral of its density in limits from *x1* to *x2*, i.e.



3. The distribution function of a continuous random variable can be expressed by the probability density:



4. The improper integral in infinite limits of the probability density of a continuous random variable is equal to 1:



Formulas of mathematical expectation and dispersion of a continuous random variable *X* have the following form:



(if the integral converges absolutely)

 (if the integral converges).

Using *D(X) = M(X2) – [M(X)]2*, we have



***Example***. Let a function *f(x)* be given:

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Find: 1) the value of the constant *а* for which the function is the probability density of some random variable; 2) the expression of the distribution function *F(x)*; 3) the probability that the random variable *X* will take on values in the segment [1; 4]; 4) draw the graph *f(x)* and *F(x).*

1) Дифференциалдық функцияның қасиеті бойынша . Біздің жағдайымызда тығыздық үш интервалда және әр интервалда тығыздық түрлі формулалармен берілген, сондықтан:



.

Сонда,  осыдан .

2) 

.

3) Интегралдық функцияны табайық. 

.

4) Дифференциалдық және интегралдық функция графиктері суретте көрсетілген.

 0 3 x

0 3 x

1

F(x)

*f(x)*

**Glossary**

**distribution function** – функция распределения

**jump of a function** – скачок функции

**step discontinuous function** – ступенчатая разрывная функция

**increment** – приращение; **distribution curve** – кривая распределения

**probability density** – плотность вероятности